QR-TLS ESPRIT for Source Localization and Frequency Estimations

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Abstract — This paper presents a subspace decomposition based algorithm for joint frequency of arrival (FOA) and direction of arrival (DOA) estimation. The proposed QR-TLS method relies on forming a Toeplitz structured matrix from the incident source signals at an antenna array. The structure of the problem favors the application of QR factorization which yields signal subspace at the expense of less number of computations. The Total least squares (TLS) solution is then applied to find the frequency and angle estimates from the signal space. The proposed QR-TLS method avoids the computations of determining the cross-correlation matrix and to apply computationally complex eigen value decomposition (EVD) or singular value decomposition (SVD) for signal subspace as required by the conventional algorithms. In addition, the proposed method can provide FOA and DOA estimates for coherent and non-coherent sources without using spatial smoothing techniques. The low computational complexity and cost favors real time applicability of the proposed method. The performance of the proposed method is shown in simulations and the results are compared with some variations in the same method.

Keywords—Joint TOA and FOA, QR, TSQR

I. INTRODUCTION
Antenna array processing techniques have extensively been utilized for direction of arrival estimation (DOA) of the incident source signals [1–7]. Estimating the incident source signal frequency jointly with the direction is of great importance in the applications of radar and wireless communication. The applications include source localization and estimating carrier frequency at the receiving end of a wireless communication system. Maximum likelihood based optimal techniques are applicable but are computationally intensive [8]. ESPRIT based joint angle and frequency estimation methods have been proposed in [9]. Multi resolution ESPRIT is used for joint angle frequency estimation in [10]. Another iterative trilinear decomposition method for joint angle and frequency estimation has been proposed in [11], [12]. The ESPRIT based methods requires EVD or SVD of the cross spectral matrix of the received data which needs high computational complexity.

In this paper we propose a subspace decomposition based algorithm that provides fast and reliable joint angle and frequency estimates. The proposed bears several advantages over the existing methods. First, the method identifies the signal space from the Toeplitz structured data matrix which relieves the complexity and computational cost to compute the cross correlation matrix. Second, the proposed method replaces the computationally intensive eigen value decomposition with the QR factorization to determine the signal subspace. Third, the integration of the TLS method with QR provides more accurate estimates in the presence of noisy measurements. Finally, the proposed method requires a small snapshot length of the received signal to estimate the FOA and DOA jointly. The low implementation complexity, computational cost and reliable performance of the proposed method favors real time applications. The method is named QR-TLS ESPRIT since it employs ESPRIT algorithm in conjugation with QR and TLS methods.

The rest of the paper is organized as follows: Section II develops the system model for joint FOA & DOA estimation. The details of the QR-TLS algorithm are described in Section III. In Section IV the simulation results are presented and compared with other variations in the same method. Finally, we conclude our paper in Section V.

II. SYSTEM MODEL
The system model assumes $K$ sources lying in a far-field region of a uniform linear array (ULA) composed of $2N + 1$ elements shown in Figure 1. The center element of the array is considered as the reference element. Each source has a carrier frequency $f_i$ and is assumed to be lying at an angle $\theta_i$ with reference to the ULA where $i = 1, 2, ..., K$.

![Uniform linear array configuration composed of 2N + 1 elements.](image)

The signal received at the $m$th element of ULA is given as

$$x_m(t) = \sum_{i=1}^{K} e^{j2\pi mf_i \sin \theta_i / c} s_i(t) + n_i(t)$$

(1)
where $x_m$ is the received signal at the $m^{th}$ element of the array, $\theta_i$ is angle of the $i^{th}$ source from the array reference, $f_i$ is the frequency of the $i^{th}$ source element, $d$ is intersensor spacing, $c$ is the speed with which the electromagnetic wave propagates and $n_i$ is additive white Gaussian noise with $\mu = 0$ and $\sigma = 1$ ($N(0, 1)$). The exponent term in (1) represents the phase shift the signal undergoes relative to the signal received at the reference element and $s_i$ denotes the $i^{th}$ source signal where $i = 1, 2, \ldots, K$.

In the proposed method we construct $(N + 1) \times 1$ sub arrays where each sub array consists of $(N + 1) \times 1$ elements. An observation matrix is constructed from a single snapshot of data received at time $t$ from each of the $(N + 1) \times 1$ sub arrays. The $[(N + 1) \times (N + 1)]$ dimension observation data matrix $X$ is formulated as

$$X = \begin{bmatrix} x_0 & x_1 & \cdots & x_N \\ x_1 & x_0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ x_N & x_{N-1} & \cdots & x_0 \end{bmatrix}$$ (2)

$$X = \begin{bmatrix} y_0 & y_1 & \ldots & y_N \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ y_N & y_{N-1} & \cdots & y_0 \end{bmatrix}$$ (3)

where $y_i = A(\theta, f) \phi(\theta, f) s_i + n_i$ for $i = 0, 1, \ldots, N$. The $A(\theta, f)$ is the array factor of the ULA and $\phi(\theta, f)$ is a diagonal matrix given as

$$A(\theta, f) = \begin{bmatrix} e^{j2\pi f_i \sin \theta_1} & e^{j2\pi f_i \sin \theta_2} & \cdots & e^{j2\pi f_i \sin \theta_N} \\ e^{j2\pi d \sin \theta_1} & e^{j2\pi d \sin \theta_2} & \cdots & e^{j2\pi d \sin \theta_N} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j2\pi d \sin \theta_1} & e^{j2\pi d \sin \theta_2} & \cdots & e^{j2\pi d \sin \theta_N} \end{bmatrix}$$ (4)

$$\phi(\theta, f) = \text{diag}(e^{-j2\pi f_i \sin \theta_1}, e^{-j2\pi f_i \sin \theta_2}, \ldots, e^{-j2\pi f_i \sin \theta_N})$$ (5)

In order to estimate the frequencies, single delayed outputs of the received signal at the antenna array from $K$ sources are added as shown in Figure 1. The delayed received signal becomes

$$y_m(t - \tau) = \sum_{i=1}^{K} e^{j2\pi m f_i \sin \theta_i / c} s_i(t - \tau) + n_i(t)$$

$$= \sum_{i=1}^{K} e^{j2\pi m f_i \sin \theta_i / c} s_i(t) e^{-j2\pi f_i \tau} + n_i(t)$$ (6)

where $y_m$ is the delayed version of the signal received signal at the $m^{th}$ element of the array.

The $[(N + 1) \times (N + 1)]$ dimension delayed observation data matrix $Y$ is given as

$$Y = \begin{bmatrix} y_0 & y_{-1} & \cdots & y_{-N} \\ y_1 & y_0 & \cdots & y_{-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_N & y_{N-1} & \cdots & y_0 \end{bmatrix}$$ (7)

$$Y = \begin{bmatrix} \Omega_0 & \Omega_1 & \ldots & \Omega_N \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_N & \Omega_{N-1} & \ldots & \Omega_0 \end{bmatrix}$$ (8)

where $\Omega_i = A(\theta, f) \phi(\theta, f) s_i \Psi(f) + n_i$ for $i = 0, 1, \ldots, N$.

$$\Psi(f) = \text{diag}(e^{-j\beta_1}, e^{-j\beta_2}, \ldots, e^{-j\beta_K})$$ (9)

where $\beta_k = 2\pi f_k \tau$ for $k = 0, 1, \ldots, K$.

The observation matrices $X$ and $Y$ are grouped into a matrix $W$ of dimension $2(N + 1) \times (N + 1)$ which will be utilized in joint FOA & DOA estimation in the following section.

$$W = \begin{bmatrix} X \\ Y \end{bmatrix}$$ (10)

III. PROPOSED QR-TLS ESPRIT METHOD

The proposed QR-TLS ESPRIT method takes the following steps in order to estimate the frequencies and angles of the multiple incident sources.

a) Frequency Estimation

The rotation matrix $\Psi(f)$ which contains the information about the frequencies of the multiple incident sources can be found by applying the straight forward least squares (LS) approach [13]. However, under noisy measurements in practical situations the LS may become inappropriate method to apply. In such case $\Psi(f)$ can be solved using the TLS method [14].

The following steps are taken in order to estimate the multiple incident source frequencies.

1. In the first step, the data matrix $W$ is decomposed into $2[(N + 1) \times (N + 1)]$ orthogonal matrix $Q$ and a $2(N + 1) \times (N + 1)$ upper triangular matrix $R$ using $QR$ factorization.

$$W = QR$$ (11)

2. The signal space of the data matrix $W$ is obtained by selecting the first $K$ columns of $Q$ which forms an orthonormal basis for the signal subspace of $W$ [14]

$$Q_z = [q(1), q(2), \ldots, q(K)]$$ (12)

where $q(i)$ denotes the $i^{th}$ column of $Q$.

3. In order to estimate the source frequencies, the rotation matrix $\Psi(f)$ is determined by partitioning the $2(N + 1) \times K$ matrix $Q_z$ into two sub-matrices $Q_{z1}$ and $Q_{z2}$ where each sub-matrix is of $(N + 1) \times K$ dimension.

$$Q_z = \begin{bmatrix} Q_{z1} \\ Q_{z2} \end{bmatrix}$$ (13)

The $\Psi(f)$ matrix is determined by applying Total Least Squares (TLS) solution as follows:

$$Y = QA$$ (14)

where $A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$ and $Q_z = \begin{bmatrix} q(1) & q(2) & \cdots & q(K) \end{bmatrix}$
1. Apply the QR decomposition on the matrix formed as

\[ [U, V] = qr(Q_h^H Q_s) = qr \left( \begin{bmatrix} Q_{s1}^H \\ Q_{s2}^H \end{bmatrix} [Q_{s1} \quad Q_{s2}] \right) \]

where \( U, V \) corresponds to \( Q \) and \( R \) of the QR factorized results respectively.

2. Partition \( U \) into \( K \times K \) sub-matrices such that

\[ U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \]  

3. Compute the eigenvalues (\( \lambda \)'s) of the matrix \( \Psi \) given as

\[ \Psi = -U_{12}U_{12}^{-1} \]  

4. Estimate the source frequencies from the eigenvalues obtained in Step 3.

\[ \hat{f}_k = \frac{\lambda_k}{2\pi f} \]

where \( \hat{f}_k \) is the estimated frequency of the \( k^{th} \) source for \( k = 0,1, \ldots, K \).

5. Estimate the elevation angle from the eigenvalues obtained in 4 and estimate the frequencies using the following expression

\[ \hat{\theta}_k = \sin^{-1} \left( \frac{\lambda_k c}{2\pi f k} \right) \]

where \( \hat{\theta}_k \) is the estimated DOA of the \( k^{th} \) source for \( k = 0,1, \ldots, K \).

IV. SIMULATION RESULTS

The performance of the proposed method with QR decomposition is assessed in simulations and benchmarked with the conventional SVD technique. Furthermore, the estimates with QR method are compared when integrated with both TLS and LS. For simulation purpose, a following test scenario is constructed:

1) Number of sensor elements of a ULA: 11
2) Number of source elements: 3
3) Source frequencies: 500 kHz, 700 kHz and 900 kHz.
4) Source angles (from the array reference): 10°, 20°, and 30°
5) Additive white Gaussian noise with mean, \( \mu = 0 \) and variance \( \sigma = 1 \).
6) The SNR of the incoming signal is varied from 0 to 50 dBs in steps of 5 dBs. For each value of SNR a Monte Carlo simulation with 100 iterations is performed and root mean square error (RMSE) in the estimated source frequencies and directions for the three sources at (500 kHz, 10°), (700 kHz, 20°), (900 kHz, 30°) is calculated from the following expression

\[ \text{RMSE} = \sqrt{\frac{1}{100} \sum_{m=1}^{100} [a_m - a_0]^2} \]

where \( a_m, a_0 \) denotes the estimated frequency/angle and actual frequency/angle respectively.

Two different snapshot lengths \( sp = 10 \) and \( sp = 40 \) are considered for simulation purpose. Figure 2 shows the RMSE of the frequency estimates against each SNR value for \( sp = 10 \). The figure shows closely following RMSE curves with both SVD and RRQR method. The figure also shows less estimation errors of the proposed method when integrated with TLS as compared to the LS. The RMSE for the angle estimates for each SNR value are shown in Figure 3. The figure shows that the estimation errors in DOA estimates with QR method are more as compared with the SVD method but the magnitude of the errors are low. The figure also confirms less estimation errors with TLS as compared to the LS.

The RMSE curves in the frequency and angle estimates for the snapshot length of \( sp = 40 \) are shown in Figure 4 and 5 respectively. Comparing the curves with the ones obtained for \( sp = 10 \), it can be seen that the RMSE curves for the present case are shifted vertically down and hence indicating less estimation errors.
The computational times of the proposed joint FOA and DOA estimation method with QR and SVD decomposition methods integrated with TLS and LS are listed in Table I for the two values of snapshot length. From the table it can be observed that the proposed QR ESPRIT method is computationally less expensive as compared with the SVD ESPRIT as it takes the least computational time. The integration of a TLS method with QR further cuts the computational time and hence favors fast performance of the proposed QR-TLS method.

V. CONCLUSIONS

The results of the paper confirm fast and reliable performance of the proposed QR-TLS ESPRIT method for joint FOA & DOA estimation. The construction of the Toeplitz matrix from the received data relieves the complexity and computational cost of determining the cross correlation matrix. The application of the QR decomposition to determine the signal space requires less computational time as compared to the EVD or SVD methods. In addition, the integration of the TLS method with QR enhances the estimation accuracy in the presence of noisy measurements. Moreover, the proposed method takes less computations to provide FOA & DOA estimates as it requires small snapshot length. In conclusion, the low computational cost, implementation complexity and small snapshot length requirement of the proposed method for joint FOA & DOA estimation makes it a favorable candidate for real time applications.

<table>
<thead>
<tr>
<th>Estimation Method</th>
<th>Computational Time (ms) for sp=10</th>
<th>Computational Time (ms) for sp=40</th>
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<tbody>
<tr>
<td>QR-TLS ESPRIT</td>
<td>16.44</td>
<td>65.4903</td>
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<tr>
<td>QR-LS ESPRIT</td>
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<td>84.9928</td>
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<td>24.32</td>
<td>99.8199</td>
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REFERENCES


